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Fermions on Lifshitz Background

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Abstract

We study a non-relativistic fermionic retarded Green's function by making use of a fermion on the Lifshitz geometry with critical exponent $z = 2$. With a natural boundary condition, respecting the symmetries of the model, the resultant retarded Green's function exhibits a number of interesting features including a flat band. We also study the finite temperature and finite chemical potential cases where the geometry is replaced by Lifshitz black hole solutions.

1 Introduction

At critical points, physics is usually described by a scale invariant model. Typically the scale invariance arises in the relativistic conformal group where we have

$$t \rightarrow \lambda t, \quad x_i \rightarrow \lambda x_i. \quad (1.1)$$

Here t is time and x_i 's are spatial directions of the space time.

We note, however, that in many physical systems the critical points are governed by dynamical scalings in which space and time scale differently. In fact spatially isotropic scale invariance is characterized by the dynamical exponent, z , as follows [1]

$$t \rightarrow \lambda^z t, \quad x_i \rightarrow \lambda x_i. \quad (1.2)$$

The corresponding critical points are known as Lifshitz fixed points.

In light of AdS/CFT correspondence [2] it is natural to seek for gravity duals of Lifshitz fixed points. Indeed gravity descriptions of Lifshitz fixed points have been considered in [3](see also [4] for an earlier work on a geometry with the Lifshitz scaling,) where a metric invariant under the scaling (1.2) was introduced. The corresponding metric is¹

$$ds^2 = L^2 \left(-\frac{dt^2}{r^{2z}} + \frac{d\bar{x}^2}{r^2} + \frac{dr^2}{r^2} \right), \quad (1.3)$$

where L is the radius of curvature². The action of the scale transformation (1.2) on the metric is given by

$$t \rightarrow \lambda^z t, \quad x_i \rightarrow \lambda x_i, \quad r \rightarrow \lambda^{-1} r. \quad (1.4)$$

As a physical application, the Lifshitz geometry has been used to provide a possible holographic description for strange metals [5]³. In this setup the Lifshitz background is probed by D-branes with non-zero gauge fields in their world volume. By choosing the dynamical critical exponent, z , the authors of [5] have been able to match the non-Fermi liquid scalings, such as linear resistivity, observed in strange metal regimes. Having found the non-Fermi liquid scalings, it is natural to study the fermionic properties of the system to explore, for example, a possibility of having a Fermi surface in the model. To do so, one needs to consider a fermion on the Lifshitz geometry to find the retarded Green's function of the corresponding dual fermionic operator via AdS/CFT correspondence.

Indeed utilizing fermions on asymptotically AdS geometries, it was shown that the AdS/CFT correspondence can holographically describe Fermi surfaces [7–13].

¹As it has been mentioned in [3], although the metric is nonsingular, it is not geodesically complete and in particular an infalling object into $r = \infty$ feels a large tidal force.

²In what follows we set $L = 1$.

³ See also [6] for drag force computations in Lifshitz geometries.

Actually to see a Fermi surface, one should look for a sharp behavior in the fermionic retarded Green's function at finite momentum and small frequencies (for a review see e.g. [14]). Moreover the spectrum of quasi-particle excitations near the Fermi surface is governed by an emergent CFT corresponding to the AdS_2 near horizon geometry of the black hole [10].

The aim of this article is to study fermions on the Lifshitz geometry which in turns can be used to study the fermionic retarded Green's function of the corresponding non-relativistic dual theory⁴. Fermions on the Lifshitz geometry or on a geometry with emergent Lifshitz geometry in its near horizon limit have been studied in [17] and [18] respectively. In these papers, with a Lorentz symmetric boundary term, the real time Green's function of the fermion has been obtained. It was shown that the resultant Green's function has real self energy. We will come back to this point in section two.

In this paper in order to find the retarded Green's function we consider the Lorentz symmetry breaking boundary condition introduced in [19]⁵. Since the corresponding boundary condition preserves rotational and scale invariances, but breaks the boost, it is more natural to impose such a boundary condition on the geometries with Lifshitz isometry. Of course it also breaks the parity which is preserved by the Lifshitz symmetry⁶.

Note that in this paper we consider the fermion as a probe. It would be interesting to extend this work to the case where the back reaction of the fermions is taken into account.

The paper is organized as follows. In the next section we study fermions on the Lifshitz geometry where we find a solution for the equation of motion with a proper boundary condition. Then using the solution we calculate the corresponding retarded Green's function where we see that the model exhibits a flat band. In section three we extend our study to the finite temperature case where we show that although the system has excited zero energy fermionic modes at low momenta, at high momenta still it has a flat band. In section four we consider charged fermions probing a charged Lifshitz black hole where we show that while with the standard boundary condition the system exhibits a Fermi surface, in the non-standard case it still has flat band. The last section is devoted to discussions.

2 Zero temperature

The aim of this section is to study fermions on the Lifshitz background which will be used to find the retarded Green's function for the corresponding fermionic dual operator in the dual non-relativistic field theory. Before going into computations,

⁴Fermions on Schrodinger spacetime has also been studied in [15, 16].

⁵Throughout this paper we refer to this boundary condition as *non-standrad* boundary condition, while we refer to that introduced in [20, 21] as *standard* boundary condition.

⁶The Lorentz symmetry breaking boundary condition has also been imposed for fermions with dipole coupling in [22] and, on the charged dilatonic black hole in [23].

it is worth to note that the Lifshitz geometry is not a solution of the pure Einstein gravity with or without cosmological constant.

In general to get the Lifshitz geometry one needs to couple the Einstein gravity to other fields. In particular the Lifshitz geometry may be obtained from gravity coupled to massive gauge fields. In the minimal case where we have only one massive gauge field, the corresponding action is given as follows

$$I = \frac{1}{16\pi G_{d+1}} \int d^{d+1}x \sqrt{g} \left(R + \Lambda - \frac{1}{4}F^2 - \frac{1}{4}m^2A^2 \right), \quad (2.1)$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. It is easy to see that, with a suitable choice of the parameters m and Λ , the model admits the Lifshitz solution (1.3) with a non-zero gauge field given by [24]

$$A_t = \sqrt{\frac{2(z-1)}{z}} \frac{1}{r^z}. \quad (2.2)$$

For this solution the parameters m and Λ are $m^2 = 4z$, $\Lambda = z^2 + (d-2)z + (d-1)^2$.

Alternatively Lifshitz metric may also be obtained as a solution of the pure gravity modified by curvature squared terms [25]. As the simplest case consider a $d+1$ dimensional gravitational action as follows

$$I = \frac{1}{2\kappa^2} \int d^{d+1}x \sqrt{-g} (R - \Lambda + \beta R^2). \quad (2.3)$$

Using the equations of motion derived from the above action one can show that the Lifshitz geometry (1.3) is a solution of the equations of motion for a suitable choice of the cosmological constant Λ and the coupling constant β that are given by

$$\Lambda = -\frac{2z^2 + (d-1)(2z+d)}{2}, \quad \beta = -\frac{1}{4\Lambda}. \quad (2.4)$$

Although we could have Lifshitz metric in arbitrary dimensions, in what follows we will consider the four dimensional Lifshitz geometry which could provide a holographic description for a three dimensional non-relativistic field theory.

2.1 Fermions on Lifshitz geometry

Let us consider a four dimensional Dirac fermion on the Lifshitz background whose action is

$$S_{\text{bulk}} = \int d^4x \sqrt{-g} i\bar{\Psi} \left[\frac{1}{2} \left(\Gamma^a \vec{D}_a - \overleftrightarrow{D}_a \Gamma^a \right) - m \right] \Psi \quad (2.5)$$

where $D^\mu = (e_\mu)^a \Gamma^\mu [\partial_a + \frac{1}{4}(\omega_{\rho\sigma})_a \Gamma^{\rho\sigma}]$, with $\Gamma^{\mu\nu} = \frac{1}{2}[\Gamma^\mu, \Gamma^\nu]$. In our notation the space-time indices are denoted by $a, b \dots$, though the tangent space indices are labeled by $\mu, \nu \dots$

Since the Lifshitz metric may be obtained from a gravity coupled to a massive gauge field, in general, the solution may also support a non-zero gauge field. Therefore one could consider a fermion that is charged under the background gauge field.

Nevertheless in what follows we will consider a neutral fermion. We will come back to charged fermions, latter, in the discussion section.

To write the equation of motion one should use the variational principle which typically comes with a proper boundary condition. It is important to note that the boundary term is not unique and indeed there are several ways to make the variational principle well defined using different boundary terms [19]. For the moment, we assume that there is a suitable boundary condition such that the variational principle will be well defined. With this assumption the equation of motion is

$$\left((e_\mu)^a \Gamma^\mu [\partial_a + \frac{1}{4} (\omega_{\rho\sigma})_a \Gamma^{\rho\sigma}] - m \right) \Psi = 0, \quad (2.6)$$

where the nonzero components of vierbeins and spin connections for the Lifshitz metric (1.3) are

$$(e_t)^a = r^z \delta^{ta}, \quad (e_i)^a = r \delta^{ia}, \quad (e_r)^a = r \delta^{ra},$$

and

$$(\omega_{tr})_a = -(\omega_{rt})_a = \frac{z}{r^z} \delta_{ta}, \quad (\omega_{ir})_a = -(\omega_{ri})_a = -\frac{1}{r} \delta_{ia}.$$

Using these expressions the equation of motion reduces to

$$\left[\Gamma^t r^z \partial_t - \left(\frac{z}{2} + 1 \right) \Gamma^r + r \Gamma^i \partial_i + r \Gamma^r \partial_r - m \right] \Psi = 0. \quad (2.7)$$

To proceed it is useful to work in the momentum space where we may set $\Psi = e^{i\omega t + ik.x} \psi(r)$. In this notation the equation of motion reads

$$ir(k.\Gamma) \psi = \left[-i\omega r^z \Gamma^t + \left(\frac{z}{2} + 1 \right) \Gamma^r - r \Gamma^r \partial_r + m \right] \psi. \quad (2.8)$$

It is also useful to act by $(D + m)$ on the first order equation of motion to find a second order differential equation which typically is easier to solve. Doing so, and using equation (2.8), one arrives at

$$\begin{aligned} (D D - m^2) \psi &= \left[r^2 \partial_r^2 - (z+2)r \partial_r + \omega^2 r^{2z} + \left(\frac{z}{2} + 1 \right) \left(\frac{z}{2} + 2 \right) - r^2 \vec{k}^2 \right. \\ &\quad \left. + i(z-1)\omega r^z \Gamma^r \Gamma^t + m \Gamma^r - m^2 \right] \psi = 0. \end{aligned} \quad (2.9)$$

In general this equation may not have analytic solutions. We note, however, that for a particular case of $m = 0$ and $z = 2$ the equation has, indeed, an analytic solution. This is the case we will consider in this paper. In this case defining $\psi_\pm = \frac{1}{2}(1 \pm \Gamma^r \Gamma^t)\psi$, one gets

$$\left[r^2 \partial_r^2 - 4r \partial_r + \omega^2 r^4 - r^2 (\vec{k}^2 \mp i\omega) + 6 \right] \psi_\pm = 0. \quad (2.10)$$

To solve the above equation we make the following change of variable

$$\psi_{\pm}(r) = r^{3/2} e^{\frac{i\omega}{2}r^2} f_{\pm}(i\omega r^2) \quad (2.11)$$

by which equation (2.10) reduces to a well known differential equation for $f_{\pm}(\xi)$

$$\frac{d^2 f_{\pm}(\xi)}{d\xi^2} + \frac{df_{\pm}(\xi)}{d\xi} + \left(\frac{\lambda_{\pm}}{\xi} + \frac{\frac{1}{4} - \mu_{\pm}^2}{\xi^2} \right) f_{\pm}(\xi) = 0, \quad (2.12)$$

where

$$\lambda_{\pm} = -\frac{k^2}{4i\omega} \pm \frac{1}{4}, \quad \mu_{\pm} = \frac{1}{4}. \quad (2.13)$$

We recognize the above equation as the hypergeometric differential equation whose solution is

$$f_{\pm}(\xi) = c_1^{\pm} \xi^{\frac{1}{2} - \mu_{\pm}} e^{-\xi} F(\alpha_{\pm}, -2\mu_{\pm} + 1, \xi) + c_2^{\pm} \xi^{\frac{1}{2} + \mu_{\pm}} e^{-\xi} F(\beta_{\pm}, 2\mu_{\pm} + 1, \xi), \quad (2.14)$$

where $F(a, b, \xi)$ is the confluent hypergeometric function, $c_{1,2}^{\pm}$ are two constant spinors and

$$\alpha_{\pm} = \frac{1}{2} - \mu_{\pm} - \lambda_{\pm}, \quad \beta_{\pm} = \frac{1}{2} + \mu_{\pm} - \lambda_{\pm}. \quad (2.15)$$

Therefore altogether we find

$$\psi_{\pm}(r) = e^{-\frac{i\omega}{2}r^2} r^2 \left[D_1^{\pm} F\left(\alpha_{\pm}, \frac{1}{2}, i\omega r^2\right) + D_2^{\pm} r F\left(\beta_{\pm}, \frac{3}{2}, i\omega r^2\right) \right], \quad (2.16)$$

with $D_1^{\pm} = (i\omega)^{\frac{1}{4}} c_1^{\pm}$, $D_2^{\pm} = (i\omega)^{\frac{3}{4}} c_2^{\pm}$.

It is important to note that so far we have solved the second order differential equation and thus the constant spinors $c_{1,2}^{\pm}$ are not independent and, indeed, restricting the above solution to be a solution of the first order equation of motion (2.8) leads to certain relations among them. More precisely one finds

$$c_2^+ = \frac{-i}{\sqrt{i\omega}} \Gamma^r(k.\Gamma) c_1^-, \quad c_2^- = \frac{-i}{\sqrt{i\omega}} \Gamma^r(k.\Gamma) c_1^+. \quad (2.17)$$

We note, also, that the solution has not been uniquely fixed yet. In fact in the context of AdS/CFT correspondence one usually imposes a boundary condition at IR. In the Euclidean case the proper boundary condition is to assume that the wave function is finite at IR. When we are dealing with the real-time AdS/CFT correspondence, the proper boundary condition is to impose an ingoing boundary condition on the wave function at the horizon [26]. In our case using the asymptotic behavior of the hypergeometric function,

$$F(a, b, \xi) \sim \frac{\Gamma(b)}{\Gamma(b-a)} (-\xi)^{-a} + \frac{\Gamma(b)}{\Gamma(a)} e^{\xi} \xi^{a-b}, \quad \text{for large } |\xi|, \quad (2.18)$$

the wave function is ingoing at “Lifshitz horizon” at $r = \infty$, if the parameters c_1^+ and c_2^+ satisfy the following relation

$$c_2^+ = -2 \frac{\Gamma(\alpha_+ + \frac{1}{2})}{\Gamma(\alpha_+)} c_1^+, \quad (2.19)$$

by which the ingoing wave function near the Lifshitz horizon behaves as follows⁷

$$\psi \sim r^2 e^{i\omega(t - \frac{r^2}{2} + \frac{k^2}{2\omega^2} \ln r)}. \quad (2.20)$$

One may wonder what “Lifshitz horizon” means! Actually the situation could be compared with that of fermions on the pure AdS case studied in, e.g. [27] where the ingoing boundary condition has been imposed at the AdS horizon at $r = \infty$. In order to understand it better one may think of this boundary condition as a limiting procedure starting from a black hole solution and then approaching the zero temperature limit, as we will do in the next section.

Alternatively to obtain the relation (2.19) and then the corresponding retarded Green’s function⁸ one may use the prescription explored in [27] where the authors presented a derivation of the real-time AdS/CFT prescription as an analytic continuation of the corresponding problem in the Euclidean signature. Indeed in our case, we have checked that, using this prescription we will arrive at the same results as those in this and the next subsections⁹.

2.2 Retarded Green’s function

In this subsection we compute the retarded Green’s function of a fermionic operator in the dual non-relativistic three dimensional field theory by making use of the solution we obtained in the previous section. One should note that, in the context of the AdS/CFT correspondence, in order to find the corresponding retarded Green’s function it is crucial to appropriately identify the source and response of the dual operator.

On the other hand the identification of the source and response depends on the boundary conditions which one imposes to get a well defined variational principle. Thus it is important to study the possible boundary terms one may add to the action to make the variational principle well defined. Therefore in what follows we will first find a proper boundary action for our model. To do so it is useful to explicitly fix our notation.

⁷Since in what follows we are interested in the low energy limit of the retarded Green’s function, the momentum will be space like, *i.e.* $k^2 \leq \omega^2$.

⁸The prescription for calculating retarded Green’s function in the context of AdS/CFT correspondence has been first considered in [26] and further studied in the literature in, e.g. [28–33].

⁹Euclidean Green’s function for fermions on the Lifshitz geometry has recently been studied in [17].

Since we have been working in a basis in which $\Gamma^r\Gamma^t$ is diagonal, we use the following representation for four dimensional gamma matrices

$$\Gamma^r = \begin{pmatrix} -\sigma^2 & 0 \\ 0 & \sigma^2 \end{pmatrix}, \Gamma^t = \begin{pmatrix} i\sigma^1 & 0 \\ 0 & i\sigma^1 \end{pmatrix}, \Gamma^1 = \begin{pmatrix} -\sigma^3 & 0 \\ 0 & -\sigma^3 \end{pmatrix}, \Gamma^2 = \begin{pmatrix} 0 & -i\sigma^2 \\ i\sigma^2 & 0 \end{pmatrix} \quad (2.21)$$

In this notation one has

$$\Psi_+ = \frac{1}{2} (1 + \Gamma^r \Gamma^t) \Psi = \text{diag}(0, 1, 1, 0) \begin{pmatrix} \Psi_1 \\ \Psi_2 \\ \Psi_3 \\ \Psi_4 \end{pmatrix} = \begin{pmatrix} 0 \\ \Psi_2 \\ \Psi_3 \\ 0 \end{pmatrix} \quad (2.22)$$

$$\Psi_- = \frac{1}{2} (1 - \Gamma^r \Gamma^t) \Psi = \text{diag}(1, 0, 0, 1) \begin{pmatrix} \Psi_1 \\ \Psi_2 \\ \Psi_3 \\ \Psi_4 \end{pmatrix} = \begin{pmatrix} \Psi_1 \\ 0 \\ 0 \\ \Psi_4 \end{pmatrix} \quad (2.23)$$

Therefore the boundary terms coming from the variation of the bulk action,

$$\delta S_{\text{bulk}} = \frac{i}{2} \int d^3x \sqrt{-h} (\bar{\Psi} \Gamma^r \delta \Psi - \delta \bar{\Psi} \Gamma^r \Psi), \quad (2.24)$$

reads

$$\delta S_{\text{bulk}} = \frac{i}{2} \int d^3x \sqrt{-h} \left[\Psi_1^\dagger \delta \Psi_1 - \Psi_2^\dagger \delta \Psi_2 - \Psi_3^\dagger \delta \Psi_3 + \Psi_4^\dagger \delta \Psi_4 - \delta \Psi_1^\dagger \Psi_1 + \delta \Psi_2^\dagger \Psi_2 + \delta \Psi_3^\dagger \Psi_3 - \delta \Psi_4^\dagger \Psi_4 \right]. \quad (2.25)$$

Since the Dirac equation is a first order differential equation we are not allowed to impose the boundary condition on all components of the spinors. Thus the aim is to add a proper boundary term such that half of the degrees of freedom do not appear on the boundary. So we will have to fix only half of the spinors.

We note, however, that the boundary terms may not be unique [19]. Of course different boundary terms lead to different physics. In our case since the dual theory is a non-relativistic field theory, one may relax the condition to have Lorentz symmetric boundary terms.

Following the suggestion of [19] it is natural to consider the following boundary term¹⁰

$$S_{\text{bdy}} = \frac{1}{2} \int d^3x \sqrt{-h} \bar{\Psi} \Gamma^1 \Gamma^2 \Psi \quad (2.26)$$

which in our notation reads

$$S_{\text{bdy}} = \frac{i}{2} \int d^3x \sqrt{-h} (\Psi_1^\dagger \Psi_3 + \Psi_2^\dagger \Psi_4 - \Psi_3^\dagger \Psi_1 - \Psi_4^\dagger \Psi_2). \quad (2.27)$$

¹⁰Note that this boundary term is different from that considered in [17].

This boundary term is invariant under rotation and scaling but breaks the boost symmetry. Of course in our model, being Lifshitz geometry, the boost symmetry has already been broken by the geometry at first place.

Adding this boundary term to the bulk action and varying the total action we arrive at

$$\begin{aligned}\delta S_{bulk} + \delta S_{bdy} &= \frac{i}{2} \int \sqrt{-h} \left[\delta(\Psi_1^\dagger + \Psi_3^\dagger)(\Psi_3 - \Psi_1) + \delta(\Psi_2^\dagger - \Psi_4^\dagger)(\Psi_2 + \Psi_4) \right. \\ &\quad \left. + (\Psi_1^\dagger - \Psi_3^\dagger)\delta(\Psi_1 + \Psi_3) + (\Psi_2^\dagger + \Psi_4^\dagger)\delta(\Psi_4 - \Psi_2) \right] \\ &= i \int \sqrt{-h} \left[-\delta\chi_1^\dagger\chi_2 + \delta\zeta_2^\dagger\zeta_1 + \chi_2^\dagger\delta\chi_1 - \zeta_1^\dagger\delta\zeta_2 \right]\end{aligned}$$

where

$$\begin{aligned}(\chi_1, \chi_2) &= \frac{1}{\sqrt{2}}(\Psi_1 + \Psi_3, \Psi_1 - \Psi_3) \\ (\zeta_1, \zeta_2) &= \frac{1}{\sqrt{2}}(\Psi_2 + \Psi_4, \Psi_2 - \Psi_4).\end{aligned}\tag{2.28}$$

Therefore we get a well defined variational principle by setting a Dirichlet boundary condition on χ_1 and ζ_2 ¹¹. As a result the source and response are given by (χ_1, ζ_2) and (χ_2, ζ_1) , respectively. The retarded Green's function is essentially a matrix which maps the source to the response. To compute the corresponding retarded Green's function it is illustrative to explicitly write the solution we have found in the previous section in components

$$\begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix} = r^2 e^{-\frac{i\omega}{2}r^2} \begin{pmatrix} D_{1\uparrow}^- F(\alpha_-; \frac{1}{2}; i\omega r^2) + D_{2\uparrow}^- r F(\beta_-; \frac{3}{2}; i\omega r^2) \\ D_{1\downarrow}^+ F(\alpha_+; \frac{1}{2}; i\omega r^2) + D_{2\downarrow}^+ r F(\beta_+; \frac{3}{2}; i\omega r^2) \\ D_{1\uparrow}^+ F(\alpha_+; \frac{1}{2}; i\omega r^2) + D_{2\uparrow}^+ r F(\beta_+; \frac{3}{2}; i\omega r^2) \\ D_{1\downarrow}^- F(\alpha_-; \frac{1}{2}; i\omega r^2) + D_{2\downarrow}^- r F(\beta_-; \frac{3}{2}; i\omega r^2) \end{pmatrix}\tag{2.29}$$

On the other hand in our basis one has

$$\Gamma^r(k \cdot \Gamma) = i \begin{pmatrix} k_1 \sigma^1 & k_2 \\ k_2 & -k_1 \sigma^1 \end{pmatrix}\tag{2.30}$$

Therefore the equation (2.17) reads

$$\begin{pmatrix} 0 \\ c_{2\downarrow}^+ \\ c_{2\uparrow}^+ \\ 0 \end{pmatrix} = \frac{1}{\sqrt{i\omega}} \begin{pmatrix} 0 \\ k_1 c_{1\uparrow}^- + k_2 c_{1\downarrow}^- \\ k_2 c_{1\uparrow}^- - k_1 c_{1\downarrow}^- \\ 0 \end{pmatrix} = \frac{1}{\sqrt{i\omega}} \begin{pmatrix} 0 \\ A c_{1\uparrow}^+ \\ A c_{1\uparrow}^+ \\ 0 \end{pmatrix}\tag{2.31}$$

¹¹If we had considered $\delta S_{bulk} - \delta S_{bdy}$ the boundary condition should have been imposed on χ_2 and ζ_1 .

where the last equality is the ingoing condition (2.19) with

$$A = -2\sqrt{i\omega} \frac{\Gamma(\alpha_+ + \frac{1}{2})}{\Gamma(\alpha_+)}. \quad (2.32)$$

By making use of this relation and utilizing the asymptotic behaviors of the solution, the retarded Green's function can be read as follows

$$G_R(k) = -\frac{1}{A^2 - k^2} \begin{pmatrix} A^2 - 2Ak_2 + k^2 & -2k_1 A \\ -2k_1 A & A^2 + 2Ak_2 + k^2 \end{pmatrix}. \quad (2.33)$$

As it is evident from the above expression for the retarded Green's function $G_{11}(\omega, k_2) = G_{22}(\omega, -k_2)$ and $\det(G_R) = -1$. The spectral function is also given by

$$\mathcal{A}(k) = -\frac{1}{\pi} \text{Im}(\text{Tr}(G_R)) = \frac{2}{\pi} \text{Im}\left(\frac{A^2 + k^2}{A^2 - k^2}\right). \quad (2.34)$$

To study different features of the retarded Green's function one may use the rotational symmetry to set $k_1 = 0$ ¹². In this case the retarded Green's function becomes diagonal. In fact taking into account that $k = k_2$ one gets

$$G_R(k_2) = -\begin{pmatrix} \frac{A-k_2}{A+k_2} & 0 \\ 0 & \frac{A+k_2}{A-k_2} \end{pmatrix}. \quad (2.35)$$

Moreover one finds

$$\mathcal{A}(k_2) = \frac{2}{\pi} \text{Im}\left(\frac{\Gamma^2(\eta + \frac{1}{2}) + \Gamma(\eta)\Gamma(\eta + 1)}{\Gamma^2(\eta + \frac{1}{2}) - \Gamma(\eta)\Gamma(\eta + 1)}\right), \quad \text{with } \eta = \frac{k_2^2}{4i\omega}. \quad (2.36)$$

The behavior of the spectral function as a function of k_2 and ω is shown in figure 1. Since the spectral function is symmetric under $k \rightarrow -k$, it is sufficient to draw the figure for positive k . As we observe the spectral function is positive for $\text{sign}(\omega) > 0$ and diverges at $\omega \rightarrow 0$. Actually by making use of the asymptotic behavior of the Gamma functions one can read the asymptotic behavior of the spectral function near $\omega = 0$. Indeed for finite k_2 one finds $\mathcal{A} \sim \frac{k_2^2}{\omega}$ showing that it has a simple pole.

To further explore the physical content of the model it is useful to study the behavior of the eigenvalues of the retarded Green's function as we approach $\omega = 0$ for fixed and finite k_2 . Indeed using the asymptotic behavior of the Gamma functions, for $k_2^2 \gg \omega$, one finds

$$\lambda_1 = \frac{k_2 - A}{k_2 + A} \approx i\frac{k_2^2}{\omega}, \quad \lambda_2 = \frac{k_2 + A}{k_2 - A} \approx -i\frac{\omega}{k_2^2}. \quad (2.37)$$

¹²Although the model has rotational symmetry, the resultant retarded Green's function (2.33) seems asymmetric with respect to exchanging k_1 and k_2 . We note that it is the artificial of our asymmetric representation of the Gamma matrices. We will back to this point latter.

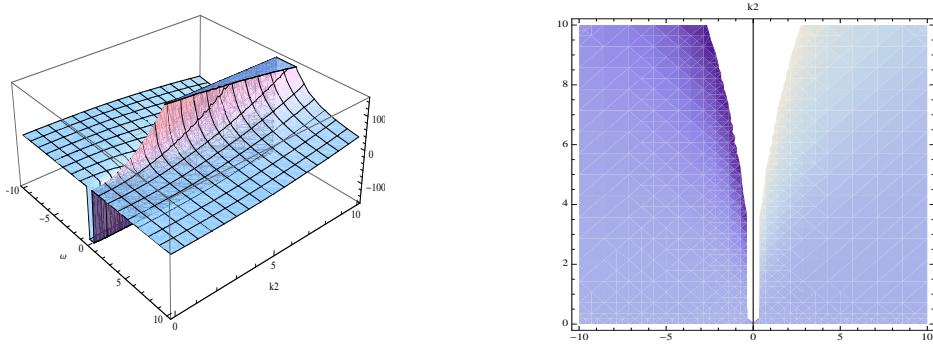


Figure 1: Three dimensional and density plots of the spectral function as a function of ω and k_2 . It is positive for $\text{sign}(\omega) > 0$ and has a pole at $\omega = 0$ for fixed k_2 .

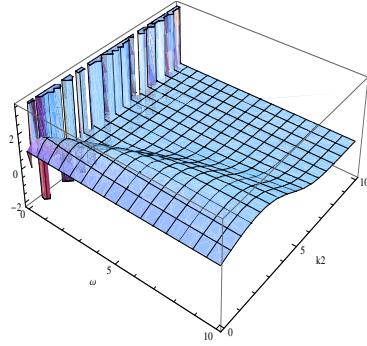


Figure 2: The real part of the flat band eigenvalue, λ_1 which shows a delta function behavior at $\omega = 0$ indicating that the imaginary part of λ_1 has a pole at $\omega = 0$.

This shows that for finite values of k_2 one of the eigenvalues, λ_1 , has a pole at $\omega = 0$. More generally one can see that the eigenvalue λ_1 has a pole at $\omega = 0$ for all values of spatial momenta. This can be seen, for example, from the behavior of the real part of the eigenvalue λ_1 where there is a delta function at $\omega = 0$ as shown in figure 2. As a result one may conclude that there are localized non-propagating excitations in the model showing that the theory exhibits a flat band.

Note that, as we have already mentioned in the introduction, the behavior of the retarded Green's function is different from that considered in [18] where by making use of a semi-holographic method it was shown that the self energy of the fermions appeared in the Green's function is real. We note, however, due to the fact we are using the Lorentz symmetry breaking boundary term the resultant Green's function has an imaginary part. Actually the situation is similar to the pure AdS case with Lorentz symmetry breaking boundary term (4.25) studied in [19]. Although in this case the bulk AdS geometry respects the Lorentz symmetry, the boundary term breaks this symmetry leading to a non-relativistic boundary theory. On the other

hand since the boundary term (4.25), up to parity, is invariant under the Lifshitz symmetry, the non-relativistic theory one gets from AdS bulk geometry has the same symmetry as if we had started with Lifshitz geometry in the bulk. Therefore one may conclude that the appearance of flat band is, indeed, the consequence of the non-relativistic feature of the dual theory.

3 Finite temperature

In this section we would like to redo the computations of the previous section for a non-relativistic theory at finite temperature. Following the general idea of gauge/gravity duality placing the dual theory at finite temperature corresponds to having a black hole in the bulk gravity. Therefore in our case we should look for a black hole solution in the asymptotic Lifshitz geometry. Actually black hole solutions in the asymptotic Lifshitz geometry have been studied in [34–37]. In particular the authors of [37] have analytically constructed a black hole which asymptotes to a vacuum Lifshitz solution with $z = 2$. The solution may be supported by different actions with different field contents, though the metric has the same form as follows

$$ds^2 = -(1 - \frac{r^2}{r_H^2}) \frac{dt^2}{r^4} + \frac{dr^2}{r^2(1 - \frac{r^2}{r_H^2})} + \frac{d\bar{x}^2}{r^2} \quad (3.1)$$

where r_H is the radius of horizon. The Hawking temperature is [37]

$$T = \frac{1}{2\pi r_H^2}. \quad (3.2)$$

3.1 Fermions on Lifshitz black hole

Following our study in the previous section we will consider a neutral massless fermion on the Lifshitz black hole given by equation (3.1). For this geometry the non-zero components of vierbeins and spin connections are

$$(e_t)^a = \frac{r^2}{\sqrt{1 - \frac{r^2}{r_H^2}}} \delta^{ta}, \quad (e_i)^a = r \delta^{ia}, \quad (e_r)^a = r \sqrt{1 - \frac{r^2}{r_H^2}} \delta^{ra},$$

and

$$(\omega_{tr})_a = -(\omega_{rt})_a = \left(\frac{2}{r^2} - \frac{1}{r_H^2} \right) \delta_{ta}, \quad (\omega_{ir})_a = -(\omega_{ri})_a = -\sqrt{\frac{1}{r^2} - \frac{1}{r_H^2}} \delta_{ia}.$$

Therefore the equation of motion for a massless fermion in this background, setting

$$\Psi = e^{i\omega t + ik \cdot x} \psi(r) = e^{i\omega t + ik \cdot x} r^2 \phi(r), \quad (3.3)$$

reduces to

$$\left[\sqrt{1 - \frac{r^2}{r_H^2}} \partial_r - \frac{r}{r_H^2 \sqrt{1 - \frac{r^2}{r_H^2}}} \left(\frac{1}{2} - i\omega r_H^2 \Gamma^r \Gamma^t \right) + i\Gamma^r k \cdot \Gamma \right] \phi(r) = 0. \quad (3.4)$$

To solve this equation it is useful to act by \not{D} on the first order equation to get a second order differential equation. In fact, defining a new variable $x = \frac{r}{r_H}$, one finds

$$(1 - x^2) \frac{d^2 \phi_{\pm}}{dx^2} - 2x \frac{d\phi_{\pm}}{dx} + \left[\nu(\nu + 1) - \frac{\mu_{\pm}^2}{1 - x^2} \right] \phi_{\pm} = 0, \quad (3.5)$$

where $\phi_{\pm} = \frac{1}{2}(1 \pm \Gamma^r \Gamma^t)\phi$, and

$$\nu = -\frac{1}{2} + ir_H \sqrt{k^2 + \omega^2 r_H^2}, \quad \mu_{\pm} = \pm \frac{1}{2} - i\omega r_H^2. \quad (3.6)$$

The resultant differential equation has a well known form whose solutions are the associated Legendre functions P and Q . Therefore the most general solution of the above equation is

$$\phi_{\pm}(r) = c_1^{\pm} P(\nu, \mu_{\pm}, x) + c_2^{\pm} Q(\nu, \mu_{\pm}, x), \quad (3.7)$$

where $c_{1,2}^{\pm}$ are constant spinors.

Of course so far we have solved the second order differential equation which its solution is not necessarily a solution of the equation of motion which is a first order differential equation. In other words the constant spinors $c_{1,2}^{\pm}$ are not independent. In fact in order to find a solution of the equation of motion one needs to plug the solution (3.7) into the equation of motion which in general leads to certain relations between the constant spinors $c_{1,2}^{\pm}$. Indeed using the recursion relations between the associated Legendre functions [38, 39],

$$\begin{aligned} (1 - x^2) \frac{dP(\nu, \mu_{\pm}, x)}{dx} &= (\nu - \mu_{\pm} + 1)(\nu + \mu_{\pm}) \sqrt{1 - x^2} P(\nu, \mu_{\pm} - 1, x) \\ &\quad + \mu_{\pm} x P(\nu, \mu_{\pm}, x) \\ &= -\sqrt{1 - x^2} P(\nu, \mu_{\pm} + 1, x) - \mu_{\pm} x P(\nu, \mu_{\pm}, x), \end{aligned} \quad (3.8)$$

one finds

$$c_{1,2}^- = ir_H \Gamma^r (k \cdot \Gamma) c_{1,2}^+. \quad (3.9)$$

In order to impose the ingoing boundary condition on the wave function at the horizon we note that at near horizon the oscillating part of the solution has the following form

$$\left(1 - \frac{r}{r_H} \right)^{\pm i\omega r_H^2}, \quad (3.10)$$

which in our notation the ingoing and outgoing waves correspond to plus and minus signs, respectively. On the other hand using the asymptotic behaviors of the associated Legendre functions near $x = 1$ one observes that the function Q has both the ingoing and the outgoing components, though the function P has only the ingoing part. Therefore in order to have a physical solution one needs to set $c_2^\pm = 0$. As a result the solution of the equation of motion of the massless fermions on the Lifshitz black hole satisfying the ingoing boundary condition is

$$\Psi^\pm = c^\pm r^2 e^{i\omega t + ik \cdot x} P \left(\nu, \mu_\pm, \frac{r}{r_H} \right), \quad (3.11)$$

with c^\pm being constant two-component spinors satisfying

$$c^- = ir_H \Gamma^r (k \cdot \Gamma) c^+. \quad (3.12)$$

3.2 Retarded Green's function

In this subsection, using the solution we just found, we will compute the retarded Green's function of a fermionic operator in the dual non-relativistic theory at finite temperature. As we mentioned in the previous section in order to compute the corresponding retarded Green's function one needs to properly identify the source and response of the dual operator which in turns depends on the boundary condition. In this section we will follow our notation in the previous section and will consider the same boundary action as that given by the equation (4.25). In this notation the source and the response of the dual operator are given by (χ_1, ζ_2) and (χ_2, ζ_1) , respectively.

In order to read the proper source and response one needs to find the asymptotic behavior of the solution as we approach the boundary. In fact by making use of the asymptotic behaviors of the associated Legendre functions (see for example [38]) one gets

$$\Psi_\pm \sim \frac{2^{\mu_\pm} \sqrt{\pi}}{\Gamma(\frac{1-\nu-\mu_\pm}{2}) \Gamma(1 + \frac{\nu-\mu_\pm}{2})} r^2 c^\pm e^{i\omega t + ik \cdot x} \equiv A_\pm r^2 c^\pm e^{i\omega t + ik \cdot x}. \quad (3.13)$$

In other words one may write

$$\Psi \sim r^2 \begin{pmatrix} A_- c_\uparrow^- \\ A_+ c_\downarrow^+ \\ A_+ c_\uparrow^+ \\ A_- c_\downarrow^- \end{pmatrix} e^{i\omega t + ik \cdot x}. \quad (3.14)$$

Note also that in our notation, equation (3.12) reads

$$\begin{pmatrix} c_\uparrow^- \\ 0 \\ 0 \\ c_\downarrow^- \end{pmatrix} = -r_H \begin{pmatrix} k_1 c_\downarrow^+ + k_2 c_\uparrow^+ \\ 0 \\ 0 \\ k_2 c_\downarrow^+ - k_1 c_\uparrow^+ \end{pmatrix} \quad (3.15)$$

Altogether with this information the retarded Green's function of the dual fermionic operator in the finite temperature non-relativistic theory is

$$G_R(k) = - \begin{pmatrix} \frac{A_+^2 + r_H^2 k^2 A_-^2 + 2r_H k_2 A_- A_+}{r_H^2 k^2 A_-^2 - A_+^2} & \frac{2r_H k_1 A_- A_+}{r_H^2 k^2 A_-^2 - A_+^2} \\ \frac{2r_H k_1 A_- A_+}{r_H^2 k^2 A_-^2 - A_+^2} & \frac{A_+^2 + r_H^2 k^2 A_-^2 - 2r_H k_2 A_- A_+}{r_H^2 k^2 A_-^2 - A_+^2} \end{pmatrix}. \quad (3.16)$$

It follows from this expression that $G_{11}(\omega, k_2) = G_{22}(\omega, -k_2)$ and $\det(G_R) = -1$. The spectral function is also given by

$$\mathcal{A}(k) = -\frac{1}{\pi} \text{Im}(\text{Tr}G_R) = \frac{2}{\pi} \text{Im}\left(\frac{r_H^2 k^2 A_-^2 + A_+^2}{r_H^2 k^2 A_-^2 - A_+^2}\right). \quad (3.17)$$

To explore different features of the retarded Green's function it is useful to use the rotational symmetry to set $k_1 = 0$. In this case the retarded Green's function reads

$$G_R(k_2) = - \begin{pmatrix} \frac{r_H k_2 A_- + A_+}{r_H k_2 A_- - A_+} & 0 \\ 0 & \frac{r_H k_2 A_- - A_+}{A_+ + r_H k_2 A_-} \end{pmatrix}. \quad (3.18)$$

Moreover for the spectral function one also finds

$$\mathcal{A}(k_2) = \frac{2}{\pi} \text{Im} \left(\frac{\frac{r_H^2 k_2^2}{4} \Gamma^2(\frac{1}{2} + X^+) \Gamma^2(\frac{1}{2} + X^-) + \Gamma^2(1 + X^+) \Gamma^2(1 + X^-)}{\frac{r_H^2 k_2^2}{4} \Gamma^2(\frac{1}{2} + X^+) \Gamma^2(\frac{1}{2} + X^-) - \Gamma^2(1 + X^+) \Gamma^2(1 + X^-)} \right), \quad (3.19)$$

with $X^\pm = \frac{i}{2}(\omega r_H^2 \pm r_H \sqrt{k_2^2 + \omega^2 r_H^2})$.

It is instructive to study the behavior of the spectral function in the small temperature limit. Physically small temperature means that we should look for the energies much higher than the temperature, *i.e.* $\frac{T}{\omega} \ll 1$. Practically one may expand the above expression for $\omega r_H^2 \gg 1$. Indeed by making use of the asymptotic behaviors of the Gamma function, up to order of $\mathcal{O}(\frac{T^2}{\omega^2})$, one arrives at

$$\begin{aligned} \mathcal{A}(k_2) &= \frac{2}{\pi} \text{Im} \left[\frac{\Gamma^2(\eta + \frac{1}{2}) + \Gamma(\eta)\Gamma(\eta + 1)}{\Gamma^2(\eta + \frac{1}{2}) - \Gamma(\eta)\Gamma(\eta + 1)} \right. \\ &\times \left. \left(1 + \frac{i\pi T}{4\omega} \frac{(4\eta - 1)\Gamma^2(\eta + \frac{1}{2})\Gamma(\eta)\Gamma(1 + \eta)}{\Gamma^4(\eta + \frac{1}{2}) - \Gamma^2(\eta)\Gamma^2(\eta + 1)} \right) \right], \end{aligned} \quad (3.20)$$

with $\eta = \frac{k_2^2}{4i\omega}$. As we see at leading order it is exactly the same expression we have found for the zero temperature case (see the equation (2.36)).

The spectral function as a function of ω and k_2 is depicted in figure 3. The plot is drawn for $r_H = \frac{2}{\sqrt{2\pi}}$ where the temperature is $T = 1/4$. To further explore the physical content of the model it is also illustrative to examine the behavior of the eigenvalues of the retarded Green's function as functions of ω and k_2 . In fact the

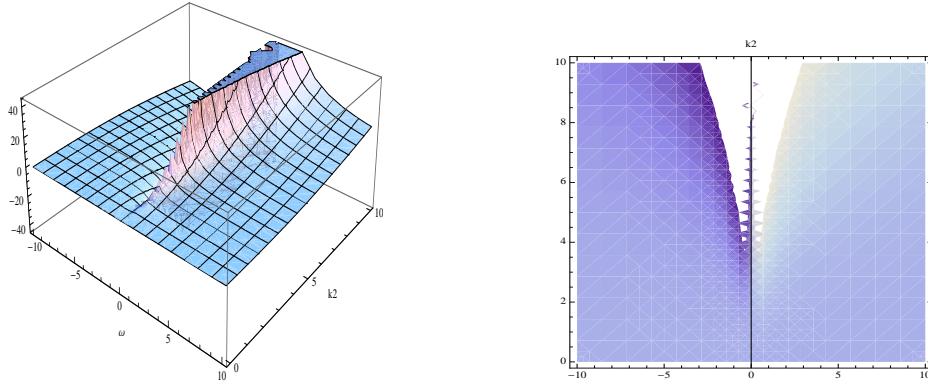


Figure 3: Three dimensional and density plots of the spectral function as a function of ω and k_2 at $r_H = \frac{2}{\sqrt{2\pi}}$. It is positive for $\text{sign}(\omega) > 0$. Note there is no pole at $\omega = 0$ for finite k_2 .

real and imaginary parts of the first eigenvalue of the retarded Green's function have been plotted in figure 4. As it is shown the imaginary part of the first eigenvalue has a pole at $\omega = 0$. This can also be seen from the delta function behavior of its real part. On the other hand it can be seen that the second eigenvalue has no pole at $\omega = 0$.

In comparison with the zero temperature case we see that the spectral function has qualitatively the same shape, though there is a small deviation in the low momentum modes. Nevertheless for high momenta it remains unchanged. Therefore the system has a flat band for high momenta.

It is worth to note that at low momenta and for low energies there are several non-trivial peaks. In fact the presence of these peaks at low energies suggest that heating up the system has excited zero energy fermionic modes at low momenta.

4 Non-zero chemical potential

It is important to mention that when one studies fermionic features of a system in condensed matter physics, usually one looks for a possibility of having a Fermi surface.

Actually in order to rigorously address this question one needs to consider a charged fermion propagating on a charged Lifshitz black hole¹³ where we could have a non-zero chemical potential. In fact as we have already mentioned in section two the Lifshitz geometry is not a solution of pure Einstein gravity. In order to find the Lifshitz solution one may couple gravity to a massive background gauge field. In

¹³ Few days after submitting our paper, another paper [40] appeared on arXiv where charged fermions on the Lifshitz geometry were studied. Of course their background is different form what we are considering in this paper for charged fermions.

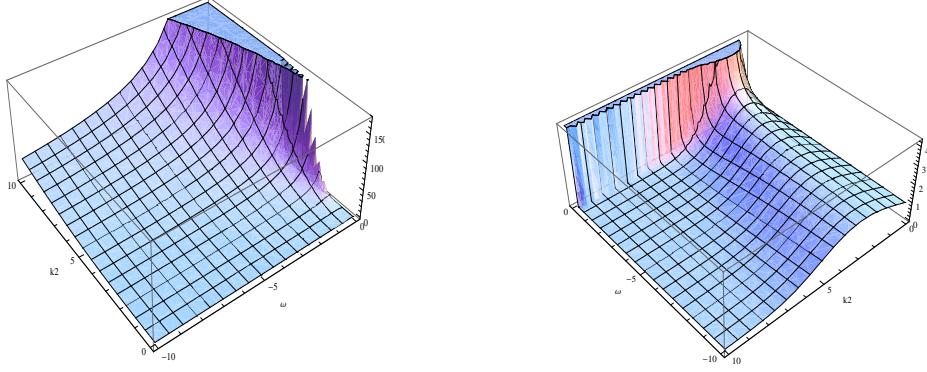


Figure 4: The imaginary (left) and real (right) parts of the flat band eigenvalue. As we see that there is a pole at $\omega = 0$ in the imaginary part of the eigenvalue which is also evident from the delta function behavior of its real part.

this case the background supports a non-zero gauge field.

We note, however, that this gauge field diverges as we approach the boundary and thus cannot play the role of chemical potential. In fact in order to have a chemical potential, another gauge field is needed [41]. Indeed the second gauge field has the proper near boundary behavior to define chemical potential. More precisely one may start with the following action [41]

$$S = \frac{1}{16\pi G_4} \int d^4x \sqrt{-g} \left[R - 2\Lambda - \frac{1}{2}(\partial\phi)^2 - \frac{1}{4}e^{\lambda_1\phi}(F^{(1)})^2 - \frac{1}{4}e^{\lambda_2\phi}(F^{(2)})^2 \right]. \quad (4.1)$$

This model admits a charged Lifshitz black hole solution with critical exponent z for the particular values of Λ, λ_1 and λ_2 as follows

$$\Lambda = -\frac{(z+1)(z+2)}{2L^2}, \quad \lambda_1 = -\frac{2}{\sqrt{z-1}}, \quad \lambda_2 = \sqrt{z-1}. \quad (4.2)$$

The corresponding black hole solution is [41]¹⁴

$$\begin{aligned} ds^2 &= -r^{2z} f dt^2 + \frac{dr^2}{r^2 f} + r^2 d\vec{x}^2, & f &= 1 - \frac{1 + r_0^{2(z+1)}}{r^{z+2}} + \frac{r_0^{2(z+1)}}{r^{2(z+1)}}, \\ e^{\sqrt{z-1}\phi} &= \frac{\kappa^2}{4z r_0^{2(z+1)}} r^{2(z-1)}, \\ A_t^{(1)} &= -\mu^{(1)} \left(1 - r^{2+z} \right), & A_t^{(2)} &= \mu^{(2)} \left(1 - \frac{1}{r^z} \right), \end{aligned} \quad (4.3)$$

¹⁴ Note that in comparison with the solution in [41] we have shifted the gauge field by constants to make sure that $g^{\mu\nu} A_\mu^{(i)} A_\nu^{(i)}$ remains finite. Moreover by a proper rescaling we have set $L = 1$ and also with a proper choice of the parameters the radius of horizon has also been set to one.

where r_0 and κ are the only remaining free parameters which determine mass and charges of the solution, and

$$\mu^{(1)} = \sqrt{\frac{2(z-1)}{z+2}} \left(\frac{\kappa^2}{4zr_0^{2(z+1)}} \right)^{\frac{1}{z-1}}, \quad \mu^{(2)} = \frac{4r_0^{2(z+1)}}{\kappa}. \quad (4.4)$$

It is, indeed, an asymptotically Lifshitz charged black hole whose Hawking temperature is

$$T = \frac{z+2}{4\pi} \left(1 - \frac{z}{z+2} r_0^{2(z+1)} \right). \quad (4.5)$$

At low energy, using the general idea of AdS/CFT correspondence, the physics is governed by near horizon modes. In this case, for example, one should study fermions on the near horizon background. At zero temperature where $r_0^{2(z+1)} = \frac{z+2}{z}$ setting

$$r - 1 = \frac{\epsilon}{(2 + 3z + z^2)\xi}, \quad t = \frac{1}{\epsilon}\tau, \quad (4.6)$$

the near horizon background can be obtained in the limit of $\epsilon \rightarrow 0$ where one finds

$$\begin{aligned} ds^2 &= \frac{-d\tau^2 + d\xi^2}{(2 + 3z + z^2)\xi^2} + dx_2^2, \quad e^\phi = \frac{\kappa^2}{4(z+2)}, \\ A_\tau^{(1)} &= \frac{\mu^{(1)}}{(z+1)\xi}, \quad A_\tau^{(2)} = \frac{z\mu^{(2)}}{(z^2 + 3z + 2)\xi}. \end{aligned} \quad (4.7)$$

As we observe the metric is in the form of $AdS_2 \times R^2$. Therefore the low energy physics is described by an emergent IR CFT. Actually the situation is very similar to the relativistic case [10]. As a result we would expect that the model exhibits a Fermi surface whose physics is governed by an IR fixed point.

4.1 Charged Fermions

Let us first consider a four dimensional charged Dirac fermion on the Lifshitz background whose action is

$$S_{\text{bulk}} = \int d^4x \sqrt{-g} i\bar{\Psi} \left[\frac{1}{2} \left(\Gamma^a \vec{D}_a - \overset{\leftarrow}{D}_a \Gamma^a \right) - m \right] \Psi. \quad (4.8)$$

Here $D = (e_\mu)^a \Gamma^\mu [\partial_a + \frac{1}{4}(\omega_{\rho\sigma})_a \Gamma^{\rho\sigma} - iqA_a^{(2)}]$, with $\Gamma^{\mu\nu} = \frac{1}{2}[\Gamma^\mu, \Gamma^\nu]$.

As we discussed in the previous sections one needs to impose a proper boundary condition to get a well defined variational principle. With a suitable boundary condition the equation of motion is

$$\left((e_\mu)^a \Gamma^\mu \left[\partial_a + \frac{1}{4}(\omega_{\rho\sigma})_a \Gamma^{\rho\sigma} - iqA_a^{(2)} \right] - m \right) \Psi = 0. \quad (4.9)$$

The above equation of motion by the choice of $\Psi = (-h)^{-1/4} e^{-i\omega t + ik \cdot x} \psi(r)$ reduces to

$$\left[r f^{1/2} \Gamma^r \partial_r - \frac{i}{r^z f^{1/2}} \left(\omega + q\mu^{(2)} \left(1 - \frac{1}{r^z} \right) \right) \Gamma^t + \frac{i}{r} \Gamma \cdot k - m \right] \psi(r) = 0. \quad (4.10)$$

As we already mentioned the low energy is governed by an emergent IR fixed point. To examine the low energy limit of the fermions one should consider the limit of $\omega \ll \mu^{(2)}$. At zero temperature, using the scaling (4.6) and in the limit of $\epsilon \rightarrow 0$ keeping ω/ϵ fixed the above equation reads

$$\left[-\Gamma^\xi \xi \partial_\xi - i\xi \Gamma^t (\tilde{\omega} + \frac{qe}{\xi}) + i\Gamma \cdot k - \frac{m}{\sqrt{2+3z+z^2}} \right] \psi(\xi) = 0, \quad (4.11)$$

where

$$e = \frac{(1+z)\mu^{(2)}}{2+3z+z^2}, \quad \tilde{\omega} = \frac{\omega}{\epsilon} = \text{finite}. \quad (4.12)$$

We recognize the above equation as a charged fermion probing the $AdS_2 \times R^2$ background (4.7), as expected. As a result one may go through the construction of [10] to express the retarded Green's function of the fermion on the Lifshitz charged black hole in terms of the retarded Green's function of AdS_2 model in small ω limit.

Here instead of doing so, one utilizes the numerical method to solve the equation of motion numerically. To proceed, it is useful to consider the following representation for four dimensional gamma matrices

$$\Gamma^r = \begin{pmatrix} -\sigma^3 & 0 \\ 0 & -\sigma^3 \end{pmatrix}, \quad \Gamma^t = \begin{pmatrix} i\sigma^1 & 0 \\ 0 & i\sigma^1 \end{pmatrix}, \quad \Gamma^1 = \begin{pmatrix} -\sigma^2 & 0 \\ 0 & \sigma^2 \end{pmatrix}, \quad \Gamma^2 = \begin{pmatrix} 0 & -i\sigma^2 \\ i\sigma^2 & 0 \end{pmatrix} \quad (4.13)$$

Due to rotational symmetry in the spatial directions we may set $k_2 = 0$. Then using the notation

$$\psi = \begin{pmatrix} \Phi_1 \\ \Phi_2 \end{pmatrix}, \quad (4.14)$$

the equation of motion (4.10) reduces to the following decoupled equations

$$\left[r f^{1/2} \partial_r - \frac{1}{r^z f^{1/2}} \left(\omega + q\mu^{(2)} \left(1 - \frac{1}{r^z} \right) \right) i\sigma^2 + m\sigma^3 - (-1)^\alpha \frac{k_1}{r} \sigma^1 \right] \Phi_\alpha = 0, \quad (4.15)$$

for $\alpha = 1, 2$. It is easy to see that

$$\Phi_\alpha \sim a_\alpha r^m \begin{pmatrix} 0 \\ 1 \end{pmatrix} + b_\alpha r^{-m} \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \text{for } r \rightarrow \infty. \quad (4.16)$$

To find the retarded Green's function, following [10], it is useful to defined $\zeta_1 = \psi_1/\psi_2$ and $\zeta_2 = \psi_3/\psi_4$ where ψ_i 's are defined via $\Phi_1 = (\psi_1, \psi_2)$, $\Phi_2 = (\psi_3, \psi_4)$. These parameters satisfy the following equations

$$rf^{1/2} \partial_r \zeta_1 + 2m\zeta_1 - \left(\frac{\Omega}{r^z f^{1/2}} + \frac{k_1}{r} \right) \zeta_1^2 = \frac{\Omega}{r^z f^{1/2}} - \frac{k_1}{r},$$

$$rf^{1/2}\partial_r\zeta_2 + 2m\zeta_2 - \left(\frac{\Omega}{r^z f^{1/2}} - \frac{k_1}{r}\right)\zeta_2^2 = \frac{\Omega}{r^z f^{1/2}} + \frac{k_1}{r}, \quad (4.17)$$

where

$$\Omega = \omega + q\mu^{(2)}\left(1 - \frac{1}{r^z}\right). \quad (4.18)$$

Using these equations the retarded Green's function is essentially given in terms of functions $G_1(k, \omega)$ and $G_2(k, \omega)$ where¹⁵

$$G_\alpha(k, \omega) = \lim_{r \rightarrow \infty} r^{2m} \zeta_\alpha, \quad \text{for } \alpha = 1, 2. \quad (4.19)$$

with the ingoing boundary condition at the horizon which in our notation it is [10]

$$\zeta_\alpha|_{\text{horizon}} = i. \quad (4.20)$$

The precise expression of the retarded Green's function in terms of G_α depends on the boundary condition one imposes to get a well defined variational principle. For example, if we impose the standard boundary condition, in our notation, the corresponding retarded Green's function is

$$G(k, \omega) = -\begin{pmatrix} G_1(k, \omega) & 0 \\ 0 & G_2(k, \omega) \end{pmatrix}. \quad (4.21)$$

Therefore the spectral function reads

$$\mathcal{A}(k, \omega) = \frac{1}{\pi} \text{Im} \left(G_1(k, \omega) + G_2(k, \omega) \right). \quad (4.22)$$

On the other hand for the boundary condition obtained by adding the boundary term (4.25) the corresponding retarded Green's function as a function of G_α is given by (see also [22])

$$G(k, \omega) = -\begin{pmatrix} \frac{2G_1(k, \omega)G_2(k, \omega)}{G_1(k, \omega) + G_2(k, \omega)} & \frac{G_1(k, \omega) - G_2(k, \omega)}{G_1(k, \omega) + G_2(k, \omega)} \\ \frac{G_1(k, \omega) - G_2(k, \omega)}{G_1(k, \omega) + G_2(k, \omega)} & \frac{-2}{G_1(k, \omega) + G_2(k, \omega)} \end{pmatrix}. \quad (4.23)$$

Thus the corresponding spectral function reads

$$\mathcal{A}(k, \omega) = 2 \text{Im} \left(\frac{G_1(k, \omega)G_2(k, \omega) - 1}{G_1(k, \omega) + G_2(k, \omega)} \right). \quad (4.24)$$

Note that in the notation we are using in this section (see (4.13)) the boundary term (4.25) reads

$$S_{bdy} = \frac{i}{2} \int d^3x \sqrt{-h} \bar{\Psi} \Gamma^1 \Gamma^2 \Psi = \frac{-i}{2} \int d^3x \sqrt{-h} (\psi_1^\dagger \psi_4 + \psi_2^\dagger \psi_3 + \psi_3^\dagger \psi_2 + \psi_4^\dagger \psi_1) \quad (4.25)$$

¹⁵Here we set $k_1 = k$.

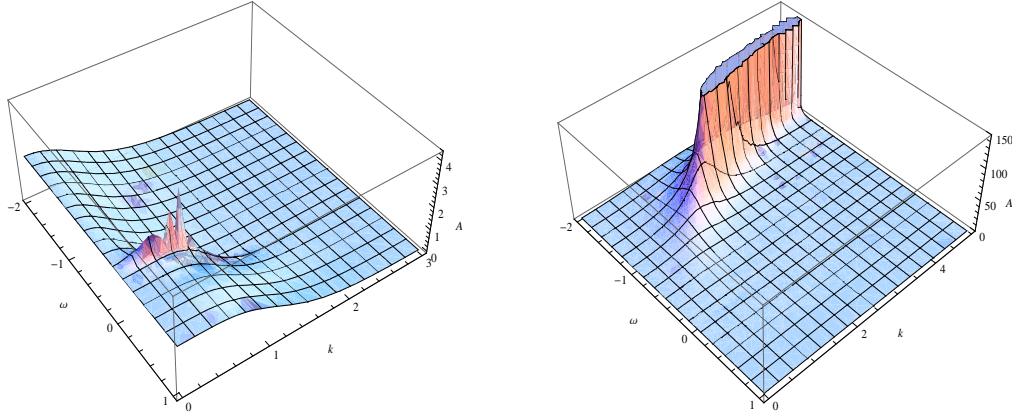


Figure 5: The behavior of spectral functions with the standard (left) and non-standard (right) boundary conditions for $z = 2$ and $T = 0$ as functions of ω and k . For the standard boundary condition the system exhibits a Fermi surface at $k_f = 0.8902$, while for the non-standard case it has a flat band.

Therefore adding this boundary term to the action results to impose the boundary condition on a combination of the different components of the fermions as follows

$$\delta S = i \int d^3x \sqrt{-h} (\delta \chi_2^\dagger \eta_2 - \delta \eta_1^\dagger \chi_1 - \chi_1^\dagger \delta \eta_1 + \eta_2^\dagger \delta \chi_2), \quad (4.26)$$

where

$$\begin{aligned} (\chi_1, \chi_2) &= \frac{1}{\sqrt{2}} (\psi_2 + \psi_4, \psi_2 - \psi_4), \\ (\eta_1, \eta_2) &= \frac{1}{\sqrt{2}} (\psi_1 + \psi_3, \psi_1 - \psi_3), \end{aligned} \quad (4.27)$$

4.2 Numerical results

Having found expressions for the retarded Green's function and the spectral function for the cases of standard and non-standard boundary conditions, it is an easy task to find their behaviors as functions of k and ω . Here to explore the physical content of the model we have plotted the spectral function of the model for both standard and non-standard boundary conditions. At zero temperature where $r_0^{2(z+1)} = (z+2)/z$ we set $m = 0$, $q\mu^{(2)} = \sqrt{3}$. For $z = 2$ the spectral function is shown in figure 5. While in the standard case the model has a Fermi surface at $k_f = 0.8902$, in the non-standard case it exhibits a flat band. Note that since the retarded Green's function is an even function of k (see equations (4.15)) we only considered $k > 0$.

It is worth to note that the model contains four free parameters which are mass m , critical exponent z , temperature T and chemical potential $\mu^{(2)}$ which always appears in the combination $q\mu^{(2)}$ where q is the charge of the fermion. Therefore it is

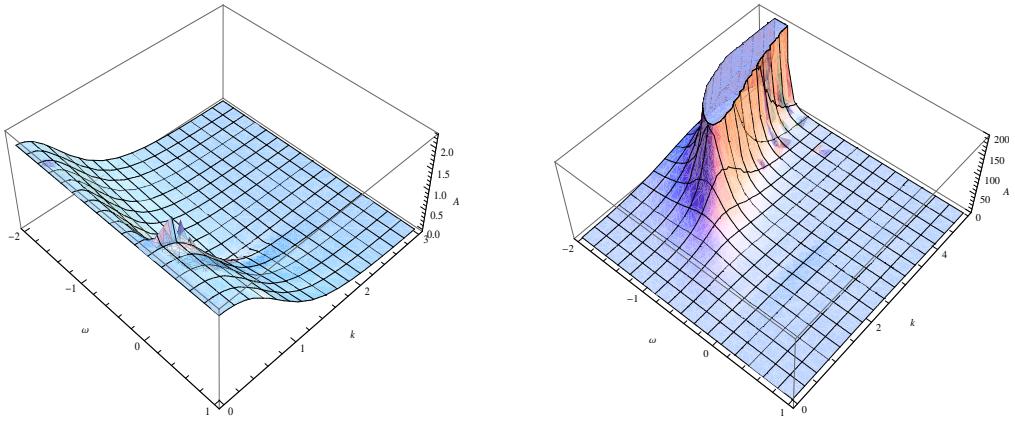


Figure 6: The behavior of spectral functions with the standard (left) and non-standard (right) boundary conditions for $z = 3$ and $T = 0$ as functions of ω and k . For the standard boundary condition the system exhibits a Fermi surface , while for the non-standard case it has a flat band.

natural to explore the physical content of the model when we vary these parameters. Actually changing the parameters we find the following behaviors.

For fixed m, T and $q\mu^{(2)}$ as we increase the critical exponent z for the standard boundary condition the sharp peak representing the Fermi surface becomes smaller and occurs at smaller k_f and eventually for large enough z it distorts the Fermi surface completely. In other words the model does not have a Fermi surface. On the other hand for the non-standard boundary condition the model still exhibits a flat band though there is a depletion in the low momentum modes (see for example figure 6 for $z = 3$).

For fixed z the dependence of the spectral function on the parameters m, T and $q\mu^{(2)}$ for both standard and non-standard cases is the same as that for $z = 1$ where we have AdS black hole solutions (see for example [10] and [19] for standard and non-standard boundary conditions, respectively).

5 Discussions

In this paper, following the general idea of AdS/CFT correspondence, we have studied retarded Green's function of a fermionic operator in a three dimensional non-relativistic field theory by making use of a massless fermion on the asymptotically Lifshitz geometry. In this paper we have mainly considered the asymptotically Lifshitz geometry with critical exponent of $z = 2$. We have considered both neutral and charged fermions.

Taking into account that the gravity on asymptotically Lifshitz backgrounds may provide holographic descriptions for strange metals [5] our studies might be useful to explore certain features of strange metals.

For neutral fermions at zero temperature where the bulk fermions propagate on the Lifshitz background the resultant retarded Green's function of the dual fermionic operator exhibits interesting behaviors. In particular we observe that the spectral function has a pole at $\omega = 0$ for all values of spatial momenta. The appearance of the pole may also be seen from the behavior of the real and imaginary parts of the eigenvalues of the corresponding retarded Green's function.

Having seen a pole at $\omega = 0$ with all values of spatial momenta shows that there are localized non-propagating excitations which in turns indicates an infinite flat band. Actually the situation is similar to that in pure AdS geometry with the Lorentz breaking boundary condition [19].

We have also considered three dimensional non-relativistic theories at finite temperature. To study the three dimensional model at finite temperature we have utilized the asymptotically Lifshitz black hole obtained in [37]. We have shown that by heating up the dual theory, although the non-zero temperature can excite low momenta zero energy modes, at high momenta there is still an infinite flat band!

An interesting feature we have seen in our model is that the spectral function is positive for $\text{sign}(\omega) > 0$. In fact unitarity requires that the spectral function to be always positive. Since in our case the retarded Green's function changes its sign and indeed is negative for $\text{sign}(\omega) < 0$, it is tempting to propose that retarded Green's function we have found for the non-relativistic model contains information for both particles and anti-particles!

We have also considered charged fermions probing a charged Lifshitz black hole. While for the standard boundary condition the model exhibits a Fermi surface for non-standard boundary condition the model still has a flat band. We have also observed that as one increases the critical exponent z for the standard boundary condition the sharp peak representing the Fermi surface becomes smaller and occurs at smaller k_f and eventually for large enough z it destroys the Fermi surface completely. In other words the model does not have a Fermi surface. On the other hand for the non-standard boundary condition the model still exhibits a flat band though there is a depletion in the low momentum modes.

It is worth to note that in order to make the variational principle well defined one could also use another boundary action as follows

$$S_{\text{bdy}} = \frac{1}{2} \int d^3x \sqrt{-h} (\bar{\Psi}_+ \Psi_- - \bar{\Psi}_- \Psi_+) = \frac{i}{2} \int d^3x \sqrt{-h} (\Psi_2^\dagger \Psi_1 + \Psi_3^\dagger \Psi_4 - \Psi_1^\dagger \Psi_2 - \Psi_4^\dagger \Psi_3). \quad (5.1)$$

With this boundary term the variation of the whole action leads to the following boundary terms

$$i \int \sqrt{-h} \left[-\delta \chi_2^\dagger \chi_1 + \delta \zeta_2^\dagger \zeta_1 + \chi_1^\dagger \delta \chi_2 - \zeta_1^\dagger \delta \zeta_2 \right], \quad (5.2)$$

where in this case the newly defined fields are given by

$$(\chi_1, \chi_2) = \frac{1}{\sqrt{2}} (\Psi_1 + \Psi_2, \Psi_1 - \Psi_2)$$

$$(\zeta_1, \zeta_2) = \frac{1}{\sqrt{2}}(\Psi_3 + \Psi_4, \Psi_3 - \Psi_4). \quad (5.3)$$

If we follow the steps we went through in the previous sections one may compute the corresponding retarded Green's function in this case. Doing so, one finds that the resultant retarded Green's functions have the same form as those in the previous sections, except the fact that the roles of k_1 and k_2 have been changed. Now in this case we could use the rotational symmetry to set $k_2 = 0$. Of course the physics remains unchanged after all.

Note added: After submitting our paper we were informed by U. Gursoy that fermionic correlation functions on Lifshitz background has recently been studied in [42]. We note, however, that the authors of this paper have used different UV boundary condition than ours.

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